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Stability analysis of a growing horizontal thermal layer subject to sudden bottom heating

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Abstract—The hydrodynamic stability considered here is formulated in terms of a two-point boundary-value, eigenvalue problem derived by assuming periodic small disturbances superimposed on a time-dependent base-state which is the quiescent growing conduction layer resulting from a step in surface heat flux. Since the local heat flux field is similar, a dimensionless similarity parameter arises. The assumed disturbance form is guided by previous stability investigations. The stiff equations are integrated to obtain the neutral stability curve with an extreme at $Gr = 202.19$, $\alpha = 1.151$, and $C_r = 0.0$. Although the postulated disturbance form allowed for time-dependent periodic oscillation, the computed disturbance wave velocity is found to be identically zero for the entire curve. Thus, it is presumed that the assumed oscillatory component of the perturbation is not time dependent, during the instability onset, at least for the calculated range ($202 \leqslant Gr \leqslant 1000$). Thus, unlike analyses for steady base-states, these results prohibit 'tracking' disturbances along constant frequency paths. Nevertheless, experimental results suggest that the wave number should decrease with increasing Gr as the neutral curve is crossed by the perturbation.

1. INTRODUCTION

As CONSIDERED here, the onset of instability is the start of the transition process from a quiescent conduction state to convective motion. This transition occurs when the motion-causing buoyancy forces which are produced by thermally generated density differences exceed the restraining viscous forces. This imbalance of forces, which is driven by heat generation at the surface, makes available an increasing amount of energy to naturally occurring disturbances as the conduction layer thickens. Results from this analysis have several implications. Transient convective flows inside heat exchangers during start-ups, air movements due to specific geo-atmospheric conditions, and behavior of thermally driven currents in large water bodies, are some of the areas where the results of this analysis can be used to a certain extent.

Phenomena associated with heated horizontal fluid were formulated by Howard [1] in a theory explaining the generation of thermals as well as the quantitative prediction for the temperature field, the Nusselt number and the duration of the preceding conductive phase. Sparrow *et al.* [2] experimentally validated the predictions of Howard by observing the generation frequency of thermals. The time of onset of instability was determined in both investigations but, according to Howard the conduction layer stability problem was not treated because of the time-dependent basic state and because of the fact that the stability of a boundary layer at the bottom of a semi-infinite region is different from the ordinary case of a finite layer.

Foster [3] analyzed the stability of a fluid layer cooled uniformly from above. The theoretical model was simplified to overcome the difficulties that arise while solving for large Rayleigh numbers. The stability equations were formulated based on infinitesimal disturbances, a method commonly used in stability analysis of convective flows. The method considers the manner in which disturbances behave in the flow, in accordance with the appropriate conservation equations. In conventional stability theory the criterion for onset of instability is determined by finding the marginal state, that is, the state where infinitesimal disturbances just start to grow. Fourier series were used to solve the equations which prohibit the solving of the problem at large Rayleigh numbers or when the flow is assumed to be two-dimensional.

The infinitesimal disturbance method has been used successfully in the past to mathematically model other flows during transition. For example, it has been used in many of the studies related to hydrodynamic stability of flows in the neighborhood of vertical, horizontal, or inclined (heated or cooled) flat plates. Its use has been exemplified by Schlichting [4] and others. Consequently, this method was selected for the present study in order to mathematically formulate the hydrodynamic stability problem of a growing one-

NOMENCLATURE

C	$C_r + iC_i$
C_i	amplification or damping ratio
C_r	disturbance wave propagation velocity, $2\pi f/\alpha$
f	disturbance frequency
F	relative error, $\phi'(0)/\phi'_{max}$
g	gravity acceleration
Gr	Grashof number, $g\beta_t\theta_c\delta^3/v^2$
i	imaginary unit
k	fluid thermal conductivity
p	local static pressure
Pr	Prandtl number, v/α_t
q''	surface heat flux
S	temperature disturbance magnitude
T	local temperature of fluid
T_s	surface temperature
T_∞	temperature at $\eta = \eta_\infty$
u	fluid velocity in the horizontal x-direction (parallel to the surface)
U	characteristic velocity, $\partial\delta/\partial\tau$
v	velocity in the vertical y-direction.

Greek symbols

α	disturbance wave number, $2\pi/\lambda$
α_t	thermal diffusivity, $k/\rho C_p$
β_t	thermal expansion coefficient
δ	characteristic length, $2\sqrt{(\alpha_t\tau)}$
η	similarity variable, y/δ
θ	temperature difference, $T - T_\infty$
θ_c	characteristic temperature, $2q''\sqrt{(\alpha_t\tau)}/k\sqrt{\pi}$
λ	perturbation wavelength
v	dynamic viscosity
ρ	fluid density
τ	time
ϕ	stream function disturbance magnitude
ψ	disturbance stream function
ω	vorticity, $\partial u/\partial y - \partial v/\partial x$.

Superscripts

$\bar{}$	average quantity
$\hat{}$	dimensional quantity of fluctuation.

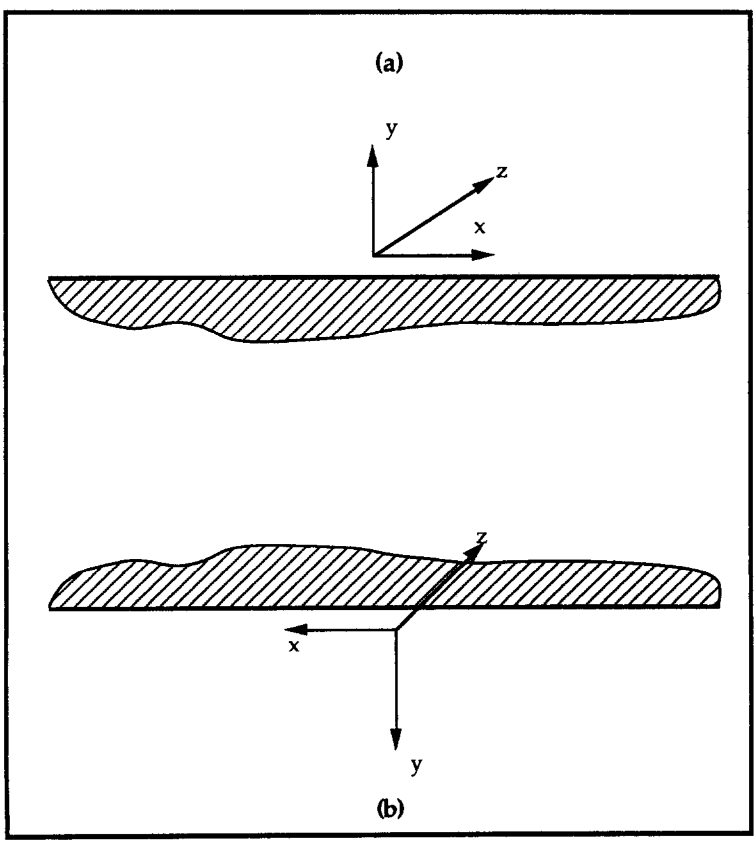

FIG. 1. The coordinate systems: (a) plate facing up $\varepsilon = +1$; (b) plate facing down $\varepsilon = -1$.

dimensional conduction layer in an extensive fluid above a suddenly heated horizontal flat plate illustrated by the top part of Fig. 1 where the shaded area represents the heat generating surface. This investigation is restricted to the task of determining theoretically whether the disturbance is amplified or decayed for a given mean flow of an incompressible fluid (water). The resulting stability equations obtained here are limited only to the onset of instability.

2. GOVERNING EQUATIONS

The governing equations are the Cartesian form of the two-dimensional continuity, momentum, and energy equations which, if the Boussinesq approximations are made, and if viscous dissipation, motion pressure, and volumetric energy generation effects are neglected, are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial \tau}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\varepsilon g\beta_t\theta \tag{3}$$

$$\frac{\partial \theta}{\partial \tau}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\alpha_t\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right) \tag{4}$$

where

$$\varepsilon=\begin{cases}+1 \text{ for heated surface facing upward}\\ -1 \text{ for heated surface facing downward.}\end{cases}$$

The case where $\varepsilon=-1$, for heated surface facing downward, represents a fluid heated uniformly from above, a naturally stable case. Conventionally, the x derivative of the y momentum equation is subtracted from the y derivative of the x momentum equation to eliminate the pressure terms. The result for a heated surface facing upward ($\varepsilon = 1$) is

$$\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right]+g\beta_t\frac{\partial \theta}{\partial x}. \tag{5}$$

The conventional vorticity function, ω, is

$$\omega=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}. \tag{6}$$

The definition of ω expressed in equation (6) reduces equation (5) to

$$\frac{\partial \omega}{\partial \tau}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\nu\left[\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right]+g\beta_t\frac{\partial \theta}{\partial x}. \tag{7}$$

Infinitesimal disturbances, $\hat{u}$, $\hat{v}$, $\hat{\omega}$ and $\hat{\theta}$, are superimposed respectively on the base-state velocity, the average vorticity $\bar{\omega}$ and the average temperature difference $\bar{\theta}$. The extensive fluid away from the growing conduction layer is motionless. If it is assumed that the heated surface is infinite in all directions parallel to the surface, and that entrainment to the surfaces is prevented, for example by building side-walls in a box shape around the perimeter of an experimental surface, then the average velocity and average vorticity of the base-state, as well as their derivatives, are reduced to zero.

Accordingly, then, equations (1), (4) and (7), are reduced to

$$\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=0 \tag{8}$$

$$\frac{\partial \hat{\omega}}{\partial \tau}+\hat{u}\frac{\partial \hat{\omega}}{\partial x}+\hat{v}\frac{\partial \hat{\omega}}{\partial y}=\nu\left[\frac{\partial^2 \hat{\omega}}{\partial x^2}+\frac{\partial^2 \hat{\omega}}{\partial y^2}\right]+g\beta_t\frac{\partial \hat{\theta}}{\partial x} \tag{9}$$

$$\frac{\partial \hat{\theta}}{\partial \tau}+\hat{u}\left(\frac{\partial \bar{\theta}}{\partial x}+\frac{\partial \hat{\theta}}{\partial x}\right)+\hat{v}\left(\frac{\partial \bar{\theta}}{\partial y}+\frac{\partial \hat{\theta}}{\partial y}\right)=\alpha_t\left(\frac{\partial^2 \hat{\theta}}{\partial x^2}+\frac{\partial^2 \hat{\theta}}{\partial y^2}\right). \tag{10}$$

Since the distribution of the base-state temperature difference, $\bar{\theta}$, is only a function of y (see Appendix), the derivative of $\bar{\theta}$ with respect to x is zero. The $\partial\bar{\theta}/\partial\tau$ and $\partial^2\bar{\theta}/\partial y^2$ terms were dropped from equation (10) as a consequence of the equality that arises when equation (4) is written in terms of average temperature and velocities ($\partial\bar{\theta}/\partial\tau=\alpha_t(\partial^2\bar{\theta}/\partial y^2)$) and recognized as the heat conduction equation for a fluid at rest. Equation (10) is then reduced to

$$\frac{\partial \hat{\theta}}{\partial \tau}+\hat{u}\frac{\partial \hat{\theta}}{\partial x}+\hat{v}\left(\frac{\partial \bar{\theta}}{\partial y}+\frac{\partial \hat{\theta}}{\partial y}\right)=\alpha_t\left(\frac{\partial^2 \hat{\theta}}{\partial x^2}+\frac{\partial^2 \hat{\theta}}{\partial y^2}\right). \tag{11}$$

Guided by the success of previous investigations reported by Schlichting [4], a disturbance stream function, $\hat{\psi}$, and a temperature disturbance function, $\hat{\theta}$, are chosen such that, for a particular disturbance with wavelength λ and frequency f, $\hat{\psi}$ and $\hat{\theta}$ are assumed to have the following forms:

$$\hat{\psi}=\hat{\phi}(y)\exp\left[i\hat{\alpha}(x-\hat{C}\tau)\right] \tag{12}$$

$$\hat{\theta}=\hat{S}(y)\exp\left[i\hat{\alpha}(x-\hat{C}\tau)\right]. \tag{13}$$

The physical wave number $\hat{\alpha}$ is related to a common disturbance wavelength such that

$$\hat{\alpha}=\frac{2\pi}{\lambda}. \tag{14}$$

Also following previous investigations, the imaginary part of the wave number is taken to be zero. The phase velocity, $\hat{C}_r$, the real part of $\hat{C}$, is defined as

$$\hat{C}_r = \frac{2\pi f}{\hat{\alpha}}. \tag{15}$$

The imaginary part of $\hat{C}$, $\hat{C}_i$, is the amplification ratio.

Since the disturbance stream function, $\hat{\psi}$, satisfies the continuity equation (8) written in terms of disturbance quantities, the vorticity, $\hat{\omega}$, can be expressed as

$$\hat{\omega} = \frac{\partial \hat{u}}{\partial y} - \frac{\partial \hat{v}}{\partial x} = \frac{\partial^2 \hat{\psi}}{\partial y^2} - \frac{\partial^2 \hat{\psi}}{\partial x^2}. \tag{16}$$

By replacing $\hat{u}$, $\hat{v}$, $\hat{\omega}$, $\hat{\theta}$, and their derivatives by their values written in terms of periodic disturbances as defined by equations (12) and (13), and by neglecting high-order (non-linear) terms, equations (8), (9), and (11) are reduced to

$$\hat{C}(\hat{\phi}'' - \hat{\alpha}^2\hat{\phi}) + \varepsilon g \beta_t \hat{S} = \frac{\mathrm{i}v}{\hat{\alpha}}(\hat{\phi}'''' - 2\hat{\alpha}^2\hat{\phi}'' + \hat{\alpha}^4\hat{\phi}) \tag{17}$$

$$\hat{C}\hat{S} - \frac{\partial \bar{\theta}}{\partial y}\hat{\phi} = -\mathrm{i}\frac{\alpha_t}{\hat{\alpha}}(\hat{\alpha}^2\hat{S} - \hat{S}''). \tag{18}$$

This pair of dimensional equations can be used to describe the stability of the evolving convective flow in the neighborhood of a suddenly heated horizontal surface.

3. STABILITY ANALYSIS

Guided by the mathematical form of the base-state temperature field solution, the following similarity variable, η, is used:

$$\eta = \frac{y}{2\sqrt{(\alpha_t \tau)}}. \tag{19}$$

Further, it follows that a characteristic length, δ, a characteristic velocity, U, and a characteristic temperature, θ_c, are defined as

$$\delta = 2\sqrt{(\alpha_t \tau)} \tag{20}$$

$$U = \frac{\partial \delta}{\partial \tau} \tag{21}$$

$$\theta_c = \frac{2q''\sqrt{(\alpha_t \tau)}}{k\sqrt{\pi}}. \tag{22}$$

By substituting non-dimensional variables into equations (17) and (18), the following Orr–Sommerfeld-like, time-dependent equations result

$$C(\phi'' - \alpha^2\phi) + \varepsilon\frac{Gr\,Pr^2}{4}S = \mathrm{i}\frac{Pr}{2\alpha}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \tag{23}$$

$$CS + \sqrt{\pi}\,\mathrm{erfc}\,(\eta)\phi = -\frac{\mathrm{i}}{2\alpha}(\alpha^2 S - S''). \tag{24}$$

Two dimensionless groups, Gr and Pr, arise in equations (23) and (24), and are defined as follows:

$$Gr = \frac{g\beta_t\theta_c\delta^3}{v^2} \tag{25}$$

$$Pr = \frac{v}{\alpha_t}. \tag{26}$$

Here, Gr is defined in terms of the time-dependent thickness of the boundary layer, and not as a function of the geometrical size of the horizontal plate. The complementary error function, erfc (η), in equation (24) arises from the substitution of $\partial\bar{\theta}/\partial y$ for $\eta = 0$ (see Appendix).

Equations (23) and (24) are coupled through the S-term of equation (23), and the system of differential equations (23) and (24), comprises a sixth-order complex-variable two-point boundary-value, eigenvalue system. Other than the eigenfunctions ϕ and S and their derivatives, there are Gr, Pr, α, C_r, and C_i, which constitute five more parameters in real space in this system. Since there are more unknowns than equations, values for some of these parameters can be specified in order to determine one set of eigenfunctions of the system, equations (23) and (24), and to compute the eigenvalues which were not specified.

Since the uniform heat flux surface is assumed to have a very small thermal capacity, the temperature disturbance amplitude, S, is not zero at the surface. Rather, the first derivative of S at the surface, $S'(0)$, must be zero since the heat flux generated by the surface is uniform. At the surface, where $\eta = 0$, the velocity disturbances, u and v, in both the x- and y-directions, must both be zero because of the no-slip boundary condition. Clearly, at large distances from the surface, where $\eta = \eta_\infty$, the disturbance quantities u, v, and θ must go to zero because the ambient is quiescent. Therefore, the boundary conditions written in terms of non-dimensional eigenfunctions for a uniform heat flux surface are

$$\phi(0) = \phi'(0) = S'(0) = \phi(\infty) = \phi'(\infty) = S(\infty) = 0. \tag{27}$$

To numerically solve the stability equations and for computational reasons, two different sets of eigenvalues can be associated with the system. The computational reasons arise from the relative variation between α and Gr. As Gr approaches its lower values, the variation of α increases and the extrapolation of a new α (guessed α) becomes less accurate. In this region, where a small variation in Gr results in a relatively high variation of α, a change in the iterative scheme used in the computation is necessary. However, at larger Gr the opposite is true. That is, a large variation of Gr results in a small variation of α. The two sets are:

(1) The set which includes the non-dimensional wave number α, the non-dimensional phase velocity C_r, and the non-dimensional degree of amplification C_i.

(2) The set which includes the non-dimensional

Table 1. The initial four iteration results

Iteration	(F_r, F_i)	α	(C_r, C_i)
0	$(6.4\times10^{-2}, 8.2\times10^{-3})$	0.2244	(0.0018, 0.0)
1	$(4.2\times10^{-4}, 5.2\times10^{-7})$	0.2235	$(5.98\times10^{-9}, 0.0)$
2	$(5.3\times10^{-8}, 5.4\times10^{-22})$	0.2229	$(7.3\times10^{-19}, 0.0)$
3	$(2.8\times10^{-8}, 0.0)$	0.2229	(0.0, 0.0)

Grashof number Gr, the non-dimensional phase velocity C_r, and the non-dimensional degree of amplification C_i.

The first set will be used during the computation of the neutral stability curve at small values of Gr and the latter set for higher values of Gr.

4. RESULTS

Due to the stiffness of the system of differential equations (23) and (24), the determination of an accurate solution was impossible using a classical finite-difference technique of integration such as Runge–Kutta. However, the technique of integration of Ascher *et al.* [5] is especially oriented for stiff systems such as the present one. The method is a multiple shooting technique for two-point boundary-value problems, but unlike other integration techniques, where the computed results are the numerical approximations of the solutions at various nodes, in Ascher *et al.*'s method the computation of the solution is carried out by a collocation which uses B-spline curve-fits of higher order (5 and 6) at Gaussian points. Ascher *et al.*'s method results in a semi-analytical solution rather than a fully numerical solution.

The integration scheme used an adaptive orthogonal collocation code (COLSYS). The code has been successfully used in previous investigations dealing with stiff differential systems which as in the study by El-Henawy [6], COLSYS was able to solve the stiff system of differential equations being investigated. An acceptable initial guess was generated by using a homotopy technique [7] which consists of incrementally shifting the equations from a state without temperature disturbances (by eliminating the coupling temperature term S) to the complete set of equations (23) and (24). When temperature disturbances are neglected, $S = 0$, the solution for the system of equations is

$$\phi = \exp(-\alpha\eta) - \exp(-\beta_1\eta) \tag{28}$$

where α is one of the eigenvalues and β_1 is defined as

$$\beta_1^2 = \alpha^2 - 2i\alpha\frac{C}{Pr}. \tag{29}$$

An iterative scheme using a Newton–Raphson method [8] was coupled to the integration scheme to compute the neutral stability curve. After the four iterative steps, which are shown in Table 1, the system converged to the eigenvalues $\alpha = 0.2229$ and $C_r = 0.0$.

To start computation for a fluid such as water, the other parameters were fixed at $Gr = 800.0$, $C_i = 0.0$ and $Pr = 6.7$. Exploring the effect of Pr (for different fluids) on the stability results was not among the objectives of this investigation, thus the computations were restricted to water which was the fluid used in experiments [9] associated with this investigation.

The relative error, F, was less than 10^{-8}, where the relative error is described as the ratio of $\phi'(0)$ to ϕ'_{max}. It can be seen from Fig. 2 that ϕ'_{max} occurs at $\eta = 3$.

Because of the coupling between S and ϕ, and the initial boundary conditions, in the final solution, the real part of the S eigenfunction and its derivative, as well as the imaginary part of the ϕ eigenfunction and its derivative ϕ', were identically zero for all values of η. To normalize the solution, $\phi'''(0)$ was set equal to unity as a constant boundary condition in order to prevent the numerical integration from converging to the trivial solution. The stations or computational nodes of the neutral stability curve were then computed in the following manner.

The B-spline coefficients of the eigenfunctions corresponding to the station (a computational node) $Gr = 800$ and $\alpha = 0.2229$ were used as the initial guess for computing a neighboring station at $Gr = 790$. Since the values of α for both $Gr = 800$ and 790 were known, a guess of α at $Gr = 780$ was made by linear extrapolation. The extrapolated guess was $\alpha = 0.2268$. Using this last value of α, $Gr = 780$, and the B-spline coefficients obtained with $Gr = 790$ as the initial guess,

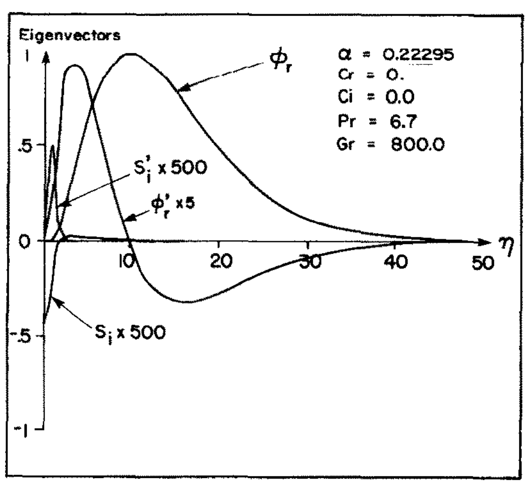

FIG. 2. Eigenvector distributions for a uniform heat flux horizontal plate at $Gr = 800.0$.

Table 2. Guessed and computed eigenvalue α for Gr ranging between 800 and 290, for the upper branch of the neutral stability curve

Gr	Guessed α	Computed α
410	3.25	3.140
460	3.33	3.400
510	3.65	3.636
560	3.84	3.851
610	4.05	4.051
660	4.23	4.237
710	4.39	4.411
760	4.58	4.576
810	4.70	4.731
860	4.87	4.880
910	5.02	5.021
960	5.15	5.156

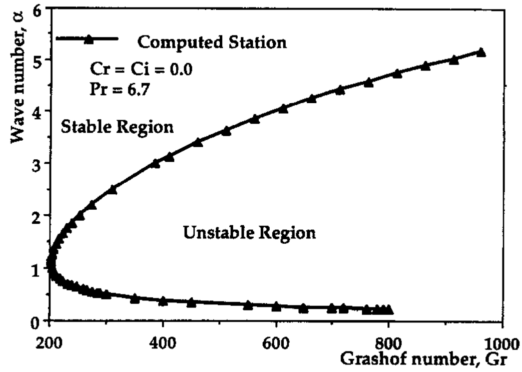

FIG. 3. Neutral stability curve for a uniform heat flux horizontal plate showing computed stations.

the new value of α converged to 0.2268. The difference between the guessed value and this result was of the order of 0.0001. A sample of the guessed as well as the computed values of α are shown in Table 2. These values are those corresponding to the upper branch portion of the neutral stability curve.

The extreme value, the computational node with the smallest Gr, and that occurring at $\alpha = 1.151$ and $Gr = 202.19$ is the nose of the neutral stability curve. This theoretical minimum value of Gr corresponding to the onset of instability of the conduction layer is in agreement with the values reported by Howard [1] and Sparrow *et al.* [2]. The critical Rayleigh numbers reported by these authors, when converted to critical Grashof numbers for water will be of the order of 200–300. Thus the results of this mathematical formulation and solving method can be perceived to be confirmatory.

For the same previously stated computational reasons associated with the different sets of eigenvalues, it was convenient to define three distinct parts of the neutral stability curve which are:

(1) the lower branch computed for Gr varying from 800 to 290,

(2) the nose branch computed for α varying from 0.55 to 3.0,

(3) the upper branch computed for Gr varying from 410 to 960.

The neutral stability curve, which was obtained by joining all the computed stations, is shown in Fig. 3. C_r, the disturbance wave propagation velocity, was found to be identically zero during the computation of all 55 points of the neutral stability curve. Since, for neutral stability ($C_i = 0$) C_r is the only time coefficient in equation (12), the above finding, C_r identically zero for all stations, points toward the presumption that the assumed oscillatory component of the perturbation is not time dependent, at least during the onset of instability, for the range of calculations presented here ($202 \leqslant Gr \leqslant 1000$).

As shown in Figs. 2 and 4, only the real part of the vorticity disturbance, ϕ, and the imaginary part of the temperature disturbance, S, were non-zero for each of the stations of the neutral stability curve. Based on this result, it appears that the mathematical modelling of the transport phenomena does not allow simultaneous computations of a non-zero solution for the temperature disturbance and an imaginary non-zero solution for the vorticity disturbance.

Some aspects of the mathematical formulation discussed in this paper were experimentally verified [9] for a growing horizontal thermal layer of water subject to sudden bottom heating. The experiments were carried out inside an insulated water tank containing an instrumented heating surface. The temperatures of both the heating surface and its adjacent water layer were monitored. The growing layer of interest was optically visualized using a Schlieren technique. The quantitative measurements of some characteristics related to the flow during its transition around the onset of instability (the variation of both the surface temperature and the disturbance size) were converted to an α–Gr plane to be compared with the theoretically obtained neutral stability curve shown on Fig. 3. Few

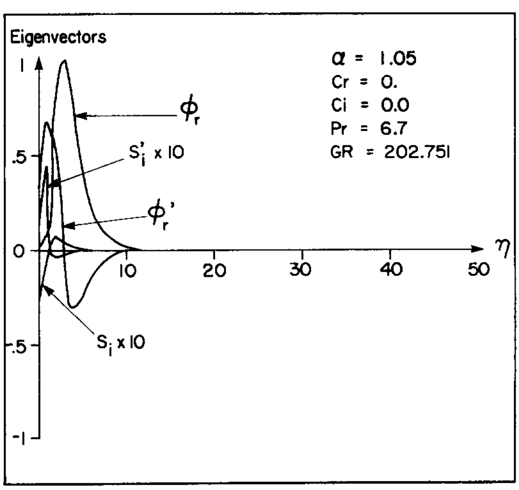

FIG. 4. Eigenvector distributions for a uniform heat flux horizontal plate at $Gr = 202.751$.

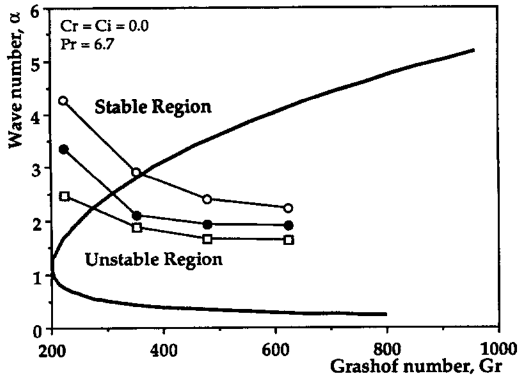

FIG. 5. Disturbance variation during flow transition for a heat flux of 1708 W m^{-2} compared to the neutral stability curve.

of the experimental results, shown on Fig. 5, reveal the evolution of three different disturbances (generated by the same heat flux) during their crossing of the neutral stability curve. These results validate the neutral stability curve portion crossed by the trajectories of the disturbances and also suggest that the wave number should decrease with increasing *Gr* as disturbances develop from stability to instability.

5. CONCLUSION

The linear stability equations have been derived and solved to evaluate the hydrodynamic stability of a growing conduction layer in an extensive fluid above a horizontal flat surface the temperature of which arises due to a step in surface heat flux. The resulting neutral stability curve indicates that the fluid above a suddenly heated horizontal surface is stable for $Gr < 202.19$. This minimum Grashof number complies with other critical numbers reported in prior theoretical and experimental investigations. Since C_r, the disturbance wave propagation velocity, was, from the computation, found to be identically zero for the entire neutral stability curve, the assumed perturbations which were suggested by previous theoretical analysis, are not time dependent during the onset of instability for the range of *Gr*'s investigated here. Experimental results, which were conducted to validate the mathematical formulation suggest that the wave number, α, should decrease with increasing *Gr* as the neutral curve is crossed by the perturbation.

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APPENDIX. DERIVATION OF THE TIME-DEPENDENT TEMPERATURE AT THE SURFACE

The following mathematical solution for the time-dependent temperature distribution in a semi-infinite solid (or non-convecting fluid) with a plane subject to a uniform heat flux was presented by Carslaw and Jaeger [10]

$$\theta(y,\tau) = T(y,\tau) - T_\infty = \frac{q''}{k}\left[\frac{\alpha_t \tau}{\pi}\exp\left(\frac{-y^2}{4\alpha_t\tau}\right) - \frac{y}{2}\operatorname{erfc}\left(\frac{y}{2\sqrt{(\alpha_t\tau)}}\right)\right]. \quad \text{(A1)}$$

The derivative of $\theta(t,\tau)$ with respect to y is

$$\frac{\partial\theta(y,\tau)}{\partial y} = \frac{-2q''}{k}\left[\frac{y}{2\sqrt{(\pi\alpha_t\tau)}}\exp\left(\frac{-y^2}{4\alpha_t\tau}\right) + \frac{y}{2}\operatorname{erfc}\left(\frac{y}{2\sqrt{(\alpha_t\tau)}}\right) + \frac{y}{2\sqrt{(\pi\alpha_t\tau)}}\exp\left(\frac{-y^2}{4\alpha_t\tau}\right)\right] \quad \text{(A2)}$$

which can be reduced to

$$\frac{\partial\theta(y,\tau)}{\partial y} = -\frac{q''}{k}\operatorname{erfc}(\eta). \quad \text{(A3)}$$

Using the chain rule, the temperature derivative with respect to η is

$$\frac{\partial\theta}{\partial\eta} = -\sqrt{\pi}\theta_c \operatorname{erfc}(\eta) \quad \text{(A4)}$$

where θ_c is defined as

$$\theta_c = \frac{2q''\sqrt{(2\alpha_t\tau)}}{k\sqrt{\pi}}. \quad \text{(A5)}$$

The above formulation of $\partial\theta/\partial\eta$ in equation (A4) supplies the complementary error function, erfc, to the stability equation.

ANALYSE DE STABILITE D'UNE COUCHE THERMIQUE HORIZONTALE SOUMISE A UN BRUSQUE CHAUFFAGE PAR DESSOUS

Résumé—La stabilité hydrodynamique considérée ici est formulée comme un problème aux valeurs propres dérivé en supposant des petites perturbations périodiques superposées à un état fondamental dépendant du temps qui est la croissance continue de la couche de conduction qui résulte d'un échelon du flux thermique en surface. Puisque le champ de flux thermique local est affine, il apparaît un paramètre de similitude sans dimension. La forme de perturbation admise est suggérée par des études antérieures de stabilité neutre avec un extremum à $Gr = 202{,}19$, $\alpha = 1{,}151$ et $C_r = 0{,}0$. La vitesse d'onde de la perturbation calculée est identiquement nulle pour la courbe entière. Il semble que la composante oscillatoire supposée de la perturbation ne dépend pas du temps pendant l'apparition de l'instabilité, au moins pour le domaine $202 \leqslant Gr \leqslant 1000$. Les résultats expérimentaux suggèrent que le nombre d'onde peut décroître quand Gr augmente à la traversée de la courbe neutre par la perturbation.

UNTERSUCHUNG DER STABILITÄT EINER WACHSENDEN WAAGERECHTEN THERMISCHEN SCHICHT BEI EINER PLÖTZLICHEN BEHEIZUNG VON UNTEN

Zusammenfassung—Die hier betrachtete hydrodynamische Stabilität wird in Form eines Zwei-Punkt-Randwert Eigenwertproblems formuliert. Dieses Problem ergibt sich durch Annahme periodischer kleiner Störungen, die einem zeitlich veränderlichen Grundzustand überlagert sind. Der Grundzustand ist die ruhig anwachsende Schicht mit Wärmeleitung, die sich infolge einer plötzlichen Änderung der Oberflächenwärmestromdichte entwickelt. Weil die örtliche Verteilung der Wärmestromdichte ähnlich ist, ergibt sich ein dimensionsloser Ähnlichkeitsparameter. Die angenommene Form der Störung wird aus früheren Stabilitätsuntersuchungen hergeleitet. Durch Integration ergibt sich die Kurve neutraler Stabilität mit einem Extremwert bei $Gr = 202{,}19$, $\alpha = 1{,}151$ und $C_r = 0{,}0$. Obwohl die geforderte Form der Störung zeitabhängige periodische Schwankungen zuläßt, ergeben die Berechnungen, daß die Wellengeschwindigkeit der Störung für die gesamte Kurve exakt gleich 0 ist. Daher wird angenommen, daß die angenommene oszillierende Komponente der Störung nicht zeitabhängig ist—dies gilt für das Einsetzen der Instabilität, wenigstens im berechneten Bereich ($202 \leqslant Gr \leqslant 1000$). Anders als bei Untersuchungen mit stationären Grundzuständen zeigen die Ergebnisse sogenannte "Trecking"-Störungen bei bestimmten Frequenzen. Die experimentellen Ergebnisse zeigen dennoch, daß die Wellenzahl bei zunehmendem Gr abnehmen sollte, falls die Neutralkurve von der Störung gekreuzt wird.

АНАЛИЗ УСТОЙЧИВОСТИ РАСТУЩЕГО ГОРИЗОНТАЛЬНОГО ТЕПЛОВОГО СЛОЯ, ПОДВЕРЖЕННОГО ДЕЙСТВИЮ ВНЕЗАПНОГО НАГРЕВА СНИЗУ

Аннотация—Для исследования гидродинамической устойчивости формулируется двухчленная кривая задача на собственные значения, полученная в предположении наложения малых периодических возмущений на иестационарное основное состояние, в котором находится неподвижный растущий кондуктивный слой, возникающий из-за градиента плотности теплового потока на поверхности. Так как поле локальных тепловых потоков является автомодельным, то возможно использовать безразмерный параметр подобия. Вид предполагаемых возмущений определяется исследованиями устойчивости, проверенными ранее. Строгие уравнения интегрируются с целью получения нейтральной кривой устойчивости с экстремальным значением при $Gr = 202{,}19$, $\alpha = 1{,}151$ и $C_r = 0{,}0$. Несмотря на то, что при предполагаемых возмущениях учитывается зависящее от времени периодическое колебание, найдено, что рассчитанная волновая скорость возмущений равна нулю для всей кривой. Таким образом, предполагается, что при возникновении неустойчивости колебательный компонент возмущения не зависит от времени по крайней мере в расчетном диапазоне ($202 \leqslant Gr \leqslant 1000$). В отличие от известного для устойчивых основных состояний, полученные результаты не позволяют обнаружить возмущений вдоль линий постоянной частоты. Тем не менее на основе экспериментальных данных можно предположить, что волновое число должно уменьшаться с ростом величины Gr, так как возмущение перечеркивает нейтральную кривую.